**Problem Sheet 1 - Submission Deadline: 16 October at 6pm (GMT) on Blackboard**

1. **[5pts]** Formulate the following problem as a linear programming problem. Write the decision variables, constraints and objective function.

Biocare makes liquid plant food for fruit and vegetables. It makes two types: Growrite (G), a high nitrogen fertiliser for green vegetables, and Tomfood (T), with a high potassium content for tomatoes, cucumbers and so on. Both types need the same basic ingredients - Ammonium Nitrate for Nitrogen (N), Phosphorus Pentoxide for Phosphorus (P), and Potassium Dioxide for Potassium (K) - but in different amounts. One litre of G requires 0.11kg of N, 0.06kg P and 0.02kg of K. One litre of T requires 0.08kg of N, 0.03kg P and 0.08kg of K. There are available each day 600kg of N, 300kg of P and 330kg of K. The selling prices per litre are £2.80 for Growrite and £3.00 for Tomfood. At these prices, Biocare can sell all it produces. Biocare wishes to maximise its daily income. How should it do so?

1. Consider the following linear programming problem.  
     
   1. **[4 pts]** Draw (by hand or using some software) the feasible set. Then determine graphically an optimal solution and indicate which constraints are active.
   2. **[4 pts]** Determine an approximate optimal solution using Matlab’s function linprog. You can either write the commands by hand or include your m-file in your submission.
2. **[5 pts]** Write the following linear programming problem in standard form.
3. **[5 pts]** Derive the dual problem of the following linear programming problem, identifying clearly the Lagrangian and the dual function.
4. **[5 pts]** Formulate the following optimization problem as a linear programming problem.
5. **[5 pts]** Let be such that . Let have full rank, and let be in the column space of . Let be the optimal solution to the linear programming problem

Prove that *.*

1. **[5 pts]** Let . For which values of is an optimal solution to the following linear programming problem?
2. **[6 pts]** Write the following linear programming problem in standard form. Then, show that this problem is infeasible if and only if there is an such that and .
3. **[6 pts]** Solve the following problem by use of the branch-and-bound method.

# Model solutions to Problem Sheet 1

*Note: The following model solutions indicate how the problem may have been solved. Alternative solutions are often also possible.*

Decision variables: the volumes (litres) of G and T to produce.

Constraints:

0.11G+ 0.08T ≤ 600

0.06G+ 0.03T ≤ 300

0.02G + 0.08T ≤ 330

G,T ≥ 0

Objective function:

Maximise 2.8G+ 3T

* 1. See OR\_PS1\_ex2\_feasible\_region.m . The feasible set is in blue.

feasible reagion of optimization problem

The following constraints are active: and

The maximal growth of the objective function is in the direction , and its level sets sets are hyperplanes with normal vector . Since the feasible region is a convex and bounded polygon, the maximum is achieved at the intersection of this polygon with the hyperplane with highest intercept at zero. This happens at the vertex defined by the two equalities  
and whose coordinates are .

* 1. See OR\_PS1\_ex2\_linprog.m . The approximate solution is .

1. We introduce the variable and write , . Then, the problem becomes

Multiplying the second inequality with -1 and moving the -6 to the righthand side, we obtain

Finally, using and as slack variables in the two inequalities, we obtain

which is in standard form.

1. Let and . Then, the Lagrangian is

The dual function is

The dual problem is

1. To formulate the optimisation problem as a linear programming problem, we need to express the objective function and constraints in linear form. The objective function is already linear. The constraint is equivalent to , which we can reformulate as a set of linear constraints by introducing new variables, as follows :

Hence, the optimisation problem is equivalent to the following linear programming problem:

1. Note that otherwise is not feasible. Let . Then, . Let . Then the vector is feasible and

Because is an optimal solution, , that is, (otherwise would be a strictly better solution).

1. Since the solution is determined by the intersection of the lines and , the solution remains optimal as long as remains between the upward pointing normals to these lines. The normal of the first line is , which gives . The normal of the first line is , which gives .
2. Let with , and let . Then, the problem is equivalent to  
   which (except for the minus sign in front of the min) is in standard form (here denotes the identity matrix). By Farkas’ lemma, this problem is infeasible iff there is a vector such that and . Splitting into three inequalities we obtain  
   which implies . Finally, defining implies that the original programming problem is infeasible iff there is such that and .

1. A first approximation to this program is and , with . Rounding up, thereby remaining feasible, we have with , , as an estimate of the optimal solution to the original program. Observe, however, that for integral values of the variables, the objective function must itself be integral. The z-value for the first approximation, , provides a lower bound for the optimal objective; consequently, the optimal objective cannot be smaller than 3. Since we have an estimate which attains the value 3, the estimate must be optimal; i.e., , with =3 .